JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **211,** 30–48 (1997)
ARTICLE NO. AY975362

Continuity Properties of ϵ-Solutions for Generalized Parametric Saddle Point Problems and Application to Hierarchical Games*

Jacqueline Morgan and Roberto Raucci

Submitted by Hélène Frankowska

Received August 10, 1995

1. INTRODUCTION

Let X, Y_1, and Y_2 be three topological spaces and f be an extended real valued function defined on $X \times Y_1 \times Y_2$. Given two multifunctions, K_1 defined on $X \times Y_2$ and non-empty valued in Y_1 and K_2 defined on $X \times Y_1$ and non-empty valued in Y_2, for any $x \in X$ let us consider the following cross constrained (also called generalized in [8]) parametric saddle point problem:

$$\mathcal{S}(x)\begin{cases} \text{find } (\bar{y}_1, \bar{y}_2) \in Y_1 \times Y_2 \text{ such that } \bar{y}_1 \in K_1(x, \bar{y}_2),\ \bar{y}_2 \in K_2(x, \bar{y}_1), \\ \text{and} \\ f(x, \bar{y}_1, \bar{y}_2) = \inf\limits_{y_1 \in K_1(x, \bar{y}_2)} f(x, y_1, \bar{y}_2) = \sup\limits_{y_2 \in K_2(x, \bar{y}_1)} f(x, \bar{y}_1, y_2). \end{cases}$$

Let $S(x)$ be the set of solutions to $\mathcal{S}(x)$.

Having in mind to give existence and approximation results for bilevel problems in which the lower level is a parametric saddle point problem, we are interested by the continuity properties of the multifunction S and more precisely we want to determine sufficient conditions on the data in order to obtain S closed graph, lower semicontinuous or open graph. Some conditions have been given in [4, 1, 7] which guarantee the multifunction S to be a closed graph but, as shown in Section 2, even for nice functions and multifunctions, the lower semicontinuity of the multifunction S is not

*This research has been supported by MURST Roma, Italy.

0022-247X/97 $25.00

guaranteed. So in this paper ϵ-saddle points will be considered and continuity properties of the multifunction $S(., \epsilon)$ defined by these approximate solutions will be studied for $\epsilon \geq 0$. More precisely, conditions of minimal character on the data will be given in a sequential setting in order to obtain $S(., \epsilon)$ lower semicontinuous, closed graph or open graph.

Note that these results will give an extension, to saddle point problems, of the continuity results obtained for approximate solutions to parametric optimization problems [14, 15, 13]. For links with the results obtained in [7] see Remarks 3.1 and 3.6.

The non-explicit constraint case and the general cross constrained case will be together considered (respectively in Sections 3 and 4). Let us note that the results of Section 4 are not a direct application of those of Section 3 but new concepts of convexity and continuity of a pair of multifunctions that have to be introduced which will be also applied to constraints defined by inequalities. In Section 5 such results will be used to give existence and approximation results for the so-called weak hierarchical saddle point problem (w-HSPP), a bilevel problem in which the lower level is a parametric saddle point problem, which corresponds to particular classes of noncooperative games with one leader and two followers ([3]),

$$\text{(w-HSPP)} \quad \begin{cases} \text{find } \bar{x} \in X \text{ such that} \\ \inf\limits_{x\in X} \sup\limits_{(y_1, y_2)\in S(x)} l(x, y_1, y_2) = \sup\limits_{(y_1, y_2)\in S(\bar{x})} l(\bar{x}, y_1, y_2), \end{cases}$$

where l is an extended real valued function defined on $X \times Y_1 \times Y_2$.

For a first introduction of the weak bilevel problems see [10] (in which such problems were called generalized Stackelberg problems).

2. PRELIMINARIES

First, we recall some definitions and continuity properties of multifunctions (see, for example, [2, 12]).

Let W_1 and W_2 be two topological spaces and G be a multifunction from W_1 to W_2. By the graph of G we indicate the following subset of $W_1 \times W_2$:

$$\text{graph } G = \{(w_1, w_2) \in W_1 \times W_2 \text{ such that } w_2 \in G(w_1)\}.$$

Moreover, let $(A_n)_n$ be a sequence of subsets of W_1 then [9]:

$u \in \operatorname{Lim\,inf}_n A_n$ if and only if there exists a sequence (u_n) converging to u in W_1 such that $u_n \in A_n$ for n sufficiently large;

$u \in \operatorname{Lim\,sup}_n A_n$ if and only if there exists a sequence (u_k) converging to u in W_1 such that $u_k \in A_{n_k}$ for a selection of integers (u_k).

DEFINITION 2.1. The multifunction G is sequentially lower semicontinuous at $w_1 \in W_1$ if for any sequence $(w_{1,n})$ converging to w_1 in W_1 we have

$$G(w_1) \subseteq \operatorname*{Lim\,inf}_n G(w_{1,n}),$$

that is, for any sequence $(w_{1,n})$ converging to w_1 in W_1 and any $w_2 \in G(w_1)$, there exists a sequence $(w_{2,n})$ converging to w_2 in W_2 such that $w_{2,n} \in G(w_{1,n})$ for n sufficiently large.

DEFINITION 2.2. The multifunction G is sequentially open graph at $w_1 \in W_1$ if for any sequence $(w_{1,n})$ converging to w_1, for any $w_2 \in G(w_1)$ and any sequence $(w_{2,n})$ converging to w_2, we have $w_{2,n} \in G(w_{1,n})$ for n sufficiently large.

DEFINITION 2.3. The multifunction G is sequentially closed graph at $w_1 \in W_1$ if for any sequence $(w_{1,n})$ converging to w_1 in W_1, we have

$$\operatorname*{Lim\,sup}_n G(w_{1,n}) \subseteq G(w_1),$$

that is, for any sequence $(w_{1,n})$ converging to w_1 in W_1 and any sequence $(w_{2,k})$ converging to w_2 in W_2 such that $w_{2,k} \in G(w_{1,n_k})$ for a selection of integers (n_k), we have $w_2 \in G(w_1)$.

In the paper, the following notations will be used:

$$v_1(x, y_2) = \inf_{y_1 \in K_1(x, y_2)} f(x, y_1, y_2); \qquad v_2(x, y_1) = \sup_{y_2 \in K_2(x, y_1)} f(x, y_1, y_2). \tag{2.1}$$

Remark 2.1. For $i = 1, 2$ let $R_i(x, y_{3-i}) = \{y_i \in K_i(x, y_{3-i})$ such that $f(x, y_1, y_2) = v_i(x, y_{3-i})\}$ and, for any $x \in X$, let $F(x)$ be the multifunction defined by

$$F(x, y_1, y_2) = R_1(x, y_2) \times R_2(x, y_1) \qquad \text{for any } (y_1, y_2) \in Y_1 \times Y_2.$$

Then the set $S(x)$ of the solutions to the problem $\mathcal{S}(x)$ is nothing but the set of fixed points of the multifunction $F(x)$.

In the following example, even if the function f and the multifunctions K_1 and K_2 are "nice," S is not lower semicontinuous.

EXAMPLE 2.1. Let $X = Y_1 = Y_2 = [0, 1]$, $f(x, y_1, y_2) = xy_1y_2$, and $K_1(x, y_2) = Y_1$, $K_2(x, y_1) = Y_2$ for any $(x, y_1, y_2) \in X \times Y_1 \times Y_2$. We have

$$S(x) = \begin{cases} [0,1] \times [0,1] & \text{if } x = 0 \\ \{0\} \times [0,1] & \text{if } x \neq 0 \end{cases}$$

and it is easy to verify that the multifunction S is not lower semicontinuous in 0.

Because the lower semicontinuity of S is essential in order to obtain existence results for multilevel problems (see Section 5), the following approximate solutions for $\mathcal{S}(x)$ will be considered:

DEFINITION 2.4 (see, for example, [6]). Let $x \in X$. The pair $(\bar{y}_1, \bar{y}_2) \in Y_1 \times Y_2$ is an ϵ-saddle point for the problem $\mathcal{S}(x)$ if it is a solution to the problem

$$\mathcal{S}(x, \epsilon) \quad \begin{cases} \text{find } (\bar{y}_1, \bar{y}_2) \in Y_1 \times Y_2 \\ \quad \text{such that } \bar{y}_1 \in K_1(x, \bar{y}_2),\ \bar{y}_2 \in K_2(x, \bar{y}_1), \\ \quad \sup\limits_{y_2 \in K_2(x, \bar{y}_1)} f(x, \bar{y}_1, y_2) - \inf\limits_{y_1 \in K_1(x, \bar{y}_2)} f(x, y_1, \bar{y}_2) \\ \quad \text{is well defined and} \\ \quad \sup\limits_{y_2 \in K_2(x, \bar{y}_1)} f(x, \bar{y}_1, y_2) - \inf\limits_{y_1 \in K_1(x, \bar{y}_2)} f(x, y_1, \bar{y}_2) \leq \epsilon, \end{cases}$$

where $a - b$ is well defined if $a - b$ is not $(+\infty) - (+\infty)$ or $(-\infty) - (-\infty)$.

We denote by $S(x, \epsilon)$ the set of solutions to the problem $\mathcal{S}(x, \epsilon)$.

Now, let us define another concept of approximate solution to the problem $\mathcal{S}(x)$ which will be useful in order to establish continuity results on $S(., \epsilon)$:

DEFINITION 2.5. Let $x \in X$. The pair $(\bar{y}_1, \bar{y}_2) \in Y_1 \times Y_2$ is a strict ϵ-saddle point to the problem $\mathcal{S}(x)$ if it is a solution to the problem

$$\tilde{\mathcal{S}}(x, \epsilon) \quad \begin{cases} \text{find } (\bar{y}_1, \bar{y}_2) \in Y_1 \times Y_2 \\ \quad \text{such that } \bar{y}_1 \in K_1(x, \bar{y}_2),\ \bar{y}_2 \in K_2(x, \bar{y}_1), \\ \quad \sup\limits_{y_2 \in K_2(x, \bar{y}_1)} f(x, \bar{y}_1, y_2) - \inf\limits_{y_1 \in K_1(x, \bar{y}_2)} f(x, y_1, \bar{y}_2) \\ \quad \text{is well defined and} \\ \quad \sup\limits_{y_2 \in K_2(x, \bar{y}_1)} f(x, \bar{y}_1, y_2) - \inf\limits_{y_1 \in K_1(x, \bar{y}_2)} f(x, y_1, \bar{y}_2) < \epsilon. \end{cases}$$

We denote by $\tilde{S}(x, \epsilon)$ the set of the strict ϵ-saddle points to the problem $\mathcal{S}(x)$, that is, the set of solutions to the problem $\tilde{\mathcal{S}}(x, \epsilon)$.

Remark 2.2. In order to obtain $v_2(x, y_1) - v_1(x, y_2)$ always well defined, we will suppose that f is marginally proper with respect to the constraints, that is,

$$\begin{cases} \text{for any } x \in X,\ y_2 \in Y_2 \text{ there exists } \bar{y}_1 \in K_1(x, y_2) \text{ such that} \\ y_2 \in K_2(x, \bar{y}_1) \text{ and } f(x, \bar{y}_1, y_2) < +\infty; \end{cases} \tag{2.2}$$

$$\begin{cases} \text{for any } x \in X,\ y_1 \in Y_1, \text{ there exists } \bar{y}_2 \in K_2(x, y_1) \text{ such that} \\ y_1 \in K_1(x, \bar{y}_2) \text{ and } f(x, y_1, \bar{y}_2) > -\infty. \end{cases} \tag{2.3}$$

If $K_1(x, y_2) = Y_1$ and $K_2(x, y_1) = Y_2$, a function f satisfying (2.2) and (2.3) will be called, as in [5], marginally proper.

Remark 2.3. As shown by Example 2.1 the multifunction $S(., \epsilon)$ can be lower semicontinuous for $\epsilon > 0$ even if S is not. In fact we have

$$S(x, \epsilon) = \begin{cases} [0,1] \times [0,1] & \text{if } x \in [0, \epsilon] \\ \left[0, \dfrac{\epsilon}{x}\right] \times [0,1] & \text{if } x \in]\epsilon, 1] \end{cases}$$

for $0 < \epsilon < 1$ and $S(x, \epsilon) = [0,1] \times [0,1]$ if $x \in [0,1]$ and $\epsilon \geq 1$.

However, let us note that the multifunction $S(., \epsilon)$ is open graph on $[0,1]$ with

$$\tilde{S}(\epsilon) = \begin{cases} \left[0, \dfrac{\epsilon}{x}\right] \times [0,1] & \text{if } x \in]\epsilon, 1] \\ [0,1] \times [0,1] & \text{if } x \in [0, \epsilon[\\ [0,1[\times [0,1] & \text{if } x = \epsilon \end{cases}$$

for $0 < \epsilon < 1$, $\tilde{S}(x, \epsilon) = [0,1] \times [0,1]$ if $x \in [0,1]$, and $\epsilon > 1$ and

$$\tilde{S}(x, 1) = \begin{cases} [0,1] \times [0,1] & \text{if } x \in [0, 1[\\ [0,1[\times [0,1] & \text{if } x = 1. \end{cases}$$

Remark 2.4. Another concept of the approximate saddle point to $S(x)$ can be defined in the following way (see [16]):

$$\hat{S}(x, \epsilon) \begin{cases} \text{find } (\bar{y}_1, \bar{y}_2) \in Y_1 \times Y_2 \\ \text{such that } \bar{y}_1 \in K_1(x, \bar{y}_2),\ \bar{y}_2 \in K_2(x, \bar{y}_1), \\ f(x, \bar{y}_1, \bar{y}_2) \leq v_1(x, \bar{y}_2) + \epsilon \text{ and } f(x, \bar{y}_1, \bar{y}_2) \geq v_2(x, \bar{y}_1) - \epsilon. \end{cases}$$

for relations with the concept given in Definition 2.4 see, for example, [6].

3. PROPERTIES OF THE MULTIFUNCTION $S(., \epsilon)$ UNDER UNEXPLICIT CONSTRAINTS

Now we suppose that for any $(x, y_1, y_2) \in X \times Y_1 \times Y_2$ we have $K_1(x, y_2) = Y_1$ and $K_2(x, y_1) = Y_2$.

In this case the saddle point problem $\mathsf{S}(x)$ becomes

$$\mathsf{S}(x) \quad \begin{cases} \text{find } (\bar{y}_1, \bar{y}_2) \in Y_1 \times Y_2 \text{ such that} \\ f(x, \bar{y}_1, \bar{y}_2) = v_1(x, \bar{y}_2) = v_2(x, \bar{y}_1), \end{cases}$$

where $v_1(x, y_2) = \inf_{y_1 \in Y_1} f(x, y_1, y_2)$ and $v_2(x, y_1) = \sup_{y_2 \in Y_2} f(x, y_1, y_2)$.

Let us recall that, for $\epsilon > 0$, the set $S(x, \epsilon)$ is non-empty iff the function $f(x, ., .)$ has a saddle-value (see, for example, [3]), that is,

$$\inf_{y_1 \in Y_1} \sup_{y_2 \in Y_2} f(x, y_1, y_2) = \sup_{y_2 \in Y_2} \inf_{y_1 \in Y_1} f(x, y_1, y_2).$$

In the following we suppose that the previous equality is satisfied and we start by giving a result about the closedness graph of the multifunction $S(., \epsilon)$.

PROPOSITION 3.1. *Let $x \in X$. If the following assumptions are satisfied,*

$$\begin{cases} \textit{for any } (y_1, y_2) \in Y_1 \times Y_2, \textit{ for any sequence } (x_n) \\ \textit{converging to } x \textit{ in } X \textit{ and any sequence } (y_{2,n}) \textit{ converging to } y_2 \\ \textit{there exists a sequence } (y_{1,n}) \textit{ such that} \\ \limsup_{n \to \infty} f(x_n, y_{1,n}, y_{2,n}) \le f(x, y_1, y_2) \end{cases} \tag{3.1}$$

$$\begin{cases} \textit{for any } (y_1, y_2) \in Y_1 \times Y_2, \textit{ for any sequence } (x_n) \\ \textit{converging to } x \textit{ in } X \textit{ and any sequence } (y_{1,n}) \textit{ converging to } y_1 \\ \textit{there exists a sequence } (y_{2,n}) \textit{ such that} \\ \liminf_{n \to \infty} f(x_n, y_{1,n}, y_{2,n}) \ge f(x, y_1, y_2) \end{cases} \tag{3.2}$$

$$\begin{cases} \textit{the function } f \textit{ is marginally proper, that is,} \\ \textit{for any } x \in X \textit{ and } y_1 \in Y_1 \\ \textit{there exists } y_2 \in Y_2 \textit{ such that } f(x, y_1, y_2) > -\infty \textit{ and} \\ \textit{for any } x \in X \textit{ and } y_2 \in Y_2 \\ \textit{there exists } y_1 \in Y_1 \textit{ such that } f(x, y_1, y_2) < +\infty, \end{cases} \tag{3.3}$$

then the multifunction $S(., \epsilon)$ is sequentially closed graph in x for any $\epsilon \ge 0$.

Proof. Let (x_n) be a sequence converging to x and $(y_{1,k}, y_{2,k})$ be a sequence converging to $(\bar{y}_1, \bar{y}_2)$ such that $(y_{1,k}, y_{2,k}) \in S(x_{n_k}, \epsilon)$ for a selection of integers (n_k). Then for any $k \in N$,

$$v_2(x_{n_k}, y_{1,k}) - v_1(x_{n_k}, y_{2,k}) \leq \epsilon \tag{3.4}$$

and from (3.3) we have

$$\liminf_{k \to \infty} v_2(x_{n_k}, y_{1,k}) - \limsup_{k \to \infty} v_1(x_{n_k}, y_{2,k}) \leq \epsilon. \tag{3.5}$$

From assumptions (3.1), (3.2), and Proposition 3.1.1 in [12], the marginal function v_1 is upper semicontinuous in $\{x\} \times Y_2$ and the marginal function v_2 is lower semicontinuous in $\{x\} \times Y_1$. Then $v_2(x, y_1) - v_1(x, y_2) \leq \epsilon$ and the proof is concluded.

Remark 3.1. In [7, Theorem 1] it is shown that if v_1 is u.s.c. and v_2 is l.s.c., the multifunction defined by approximate solutions related to our concept is closed graph. Differently, in Proposition 3.1, we gave sufficient conditions for the closedness, not on marginal functions but explicitly on the data of the problem.

Moreover Proposition 3.1 corresponds to a slight generalization of Theorem 3.10 in [1] in which the following assumptions are considered in order to obtain also the convergence of the values:

$$\begin{cases} \text{for any } (y_1, y_2) \in Y_1 \times Y_2, \text{ for any sequence } (x_n) \text{ converging to } x \text{ in } X, \\ \text{and any sequence } (y_{2,n}) \text{ converging to } y_2, \text{ there exists a sequence} \\ (y_{1,n}) \text{ converging to } y_1 \text{ such that } \limsup\limits_{n \to \infty} f(x_n, y_{1,n}, y_{2,n}) \leq f(x, y_1, y_2) \end{cases} \tag{3.1e}$$

$$\begin{cases} \text{for any } (y_1, y_2) \in Y_1 \times Y_2, \text{ for any sequence } (x_n) \text{ converging to } x \text{ in } X, \\ \text{and any sequence } (y_{1,n}) \text{ converging to } y_1, \text{ there exists a sequence} \\ (y_{2,n}) \text{ converging to } y_2 \text{ such that } \liminf\limits_{n \to \infty} f(x_n, y_{1,n}, y_{2,n}) \geq f(x, y_1, y_2). \end{cases} \tag{3.2e}$$

For example, the function $f\colon [0,1]^3 \to R$ defined by

$$f(x, y_1, y_2) = \begin{cases} y_1 + y_2 & \text{if } (y_1, y_2) \neq (0,0) \\ \frac{1}{2} & \text{if } (y_1, y_2) = (0,0) \end{cases}$$

satisfies conditions (3.1e) and (3.2) but not condition (3.2e).

Finally we note that, as shown by the following example, the assumption (3.3) is an essential one.

EXAMPLE 3.1. Let f: $[-1,1]^3 \to R \cup \{+\infty, -\infty\}$ defined by

$$f(x, y_1, y_2) = \begin{cases} \dfrac{1}{|x| + |y_2|} & \text{if } (x, y_2) \neq (0,0) \\ +\infty & \text{if } (x, y_2) = (0,0). \end{cases}$$

The function f is continuous so conditions (3.1) and (3.2) are satisfied but $f(0, y_1, 0) = +\infty$ for any $y_1 \in [-1,1]$ and (3.3) is not satisfied for $x = 0$.

Now, let (x_n) be a sequence in $[-1,1] - \{0\}$ converging to 0 and let $y_{2,n} = x_n^3$. We can prove $(0,0) \notin S(0, \epsilon)$ and $(0,0) \in \operatorname{Lim\,sup} S(x_n, \epsilon)$ for any $\epsilon > 0$. So the multifunction $S(., \epsilon)$ is not closed graph in 0.

Let us note that the assumption (3.3) can be weakened in the following way: there exists a neighbourhood U of x such that for any $u \in U$, for any $y_1 \in Y_1$, there exists $y_2 \in Y_2$ such that $f(u, y_1, y_2) > -\infty$ and for any $u \in U$, for any $y_2 \in Y_2$, there exists $y_1 \in Y_1$ such that $f(u, y_1, y_2) < +\infty$.

Remark 3.2. It can easily be proved that under the assumptions of Proposition 3.1 we have also $\operatorname{Lim\,sup}_n S(x_n, \epsilon_n) \subseteq S(x)$ for any sequence (x_n) converging to x and any sequence (ϵ_n) converging to 0.

Now, let us give a lower semicontinuity result about the multifunction $\tilde{S}(., \epsilon)$ defined in Definition 2.5.

PROPOSITION 3.2. *Let $x \in X$. If the following assumptions are satisfied,*

$$\begin{cases} \textit{for any sequence } (x_n) \textit{ converging to } x \textit{ and any } y_2 \in Y_2 \textit{ there exists a} \\ \textit{sequence } (\bar{y}_{2,n}) \textit{ converging to } y_2 \textit{ such that for any sequence } (y_{1,n}) \textit{ in } Y_1 \\ \textit{there exists } \bar{y}_1 \in Y_1 \textit{ satisfying } \liminf\limits_{n\to\infty} f(x_n, y_{1,n}, \bar{y}_{2,n}) \geq f(x, \bar{y}_1, y_2) \end{cases} \tag{3.6}$$

$$\begin{cases} \textit{for any sequence } (x_n) \textit{ converging to } x \textit{ and any } y_1 \in Y_1 \textit{ there exists a} \\ \textit{sequence } (\bar{y}_{1,n}) \textit{ converging to } y_1 \textit{ such that for any sequence } (y_{2,n}) \textit{ in } Y_2 \\ \textit{there exists } \bar{y}_2 \in Y_2 \textit{ satisfying } \limsup\limits_{n\to\infty} f(x_n, \bar{y}_{1,n}, y_{2,n}) \leq f(x, y_1, \bar{y}_2), \end{cases} \tag{3.7}$$

then the multifunction $\tilde{S}(., \epsilon)$ is sequentially lower semicontinuous at x for any $\epsilon > 0$.

Proof. First of all let us prove

$$\begin{cases} \text{for any sequence } (x_n) \text{ converging to } x \text{ and any } y_2 \in Y_2 \text{ there exists a} \\ \text{sequence } (\bar{y}_{2,n}) \text{ converging to } y_2 \text{ such that} \\ \liminf_{n\to\infty} v_1(x_n, \bar{y}_{2,n}) \geq v_1(x, y_2). \end{cases} \tag{3.8}$$

In fact, let (x_n) converging to x, $y_2 \in Y_2$ and $(\bar{y}_{2,n})$ be the sequence defined in (3.6). There exists a minimizing sequence $(\bar{y}_{1,n})$ such that

$$\liminf_{n\to\infty} f(x_n, \bar{y}_{1,n}, \bar{y}_{2,n}) \leq \liminf_{n\to\infty} v_1(x_n, \bar{y}_{2,n}).$$

From (3.6) there exists $y_1 \in Y_1$ such that

$$\liminf_{n\to\infty} f(x_n, \bar{y}_{1,n}, \bar{y}_{2,n}) \geq f(x, \bar{y}_1, \bar{y}_2).$$

But $f(x, \bar{y}_1, y_2) \geq v_1(x, y_2)$ so we can deduce (3.8).

Analogously we can prove

$$\begin{cases} \text{for any sequence } (x_n) \text{ converging to } x \text{ and any } y_1 \in Y_1 \\ \text{there exists a sequence } (\bar{y}_{1,n}) \text{ converging to } y_1 \text{ such that} \\ \limsup_{n\to\infty} v_2(x_n, \bar{y}_{1,n}) \leq v_2(x, y_1). \end{cases} \tag{3.9}$$

Now, let us prove that $\tilde{S}(., \epsilon)$ is lower semicontinuous at $x \in X$ for any $\epsilon > 0$. In fact let (x_n) converging to x and $(y_1, y_2) \in \tilde{S}(x, \epsilon)$. From (3.8) and (3.9) there exist two sequences $(\bar{y}_{1,n})$ and $(\bar{y}_{2,n})$ converging respectively to y_1 and y_2 such that

$$\limsup_{n\to\infty} v_2(x_n, \bar{y}_{1,n}) - \liminf_{n\to\infty} v_1(x_n, \bar{y}_{2,n}) \leq v_2(x, y_1) - v_1(x, y_2) < \epsilon.$$

Let $\alpha = (\epsilon - h)/2$ with

$$h = \limsup_{n\to\infty} v_2(x_n, \bar{y}_{1,n}) - \liminf_{n\to\infty} v_1(x_n, \bar{y}_{2,n}).$$

There exists $n_1(\epsilon)$ such that, for $n \geq n_1(\epsilon)$,

$$v_2(x_n, \bar{y}_{1,n}) < \limsup_{n\to\infty} v_2(x_n, \bar{y}_{1,n}) + \alpha$$

and there exists $n_2(\epsilon)$ such that, for $n \geq n_2(\epsilon)$,

$$v_1(x_n, \bar{y}_{2,n}) > \liminf_{n\to\infty} v_1(x_n, \bar{y}_{2,n}) - \alpha;$$

then, for $n \geq n_0(\epsilon) = \max\{n_1(\epsilon), n_2(\epsilon)\}$ we have

$$\begin{aligned} & v_2(x_n, \bar{y}_{1,n}) - v_1(x_n, \bar{y}_{2,n}) \\ & \quad < \limsup_{n \to \infty} v_2(x_n, \bar{y}_{1,n}) - \liminf_{n \to \infty} v_1(x_n, \bar{y}_{2,n}) + 2\alpha \\ & \quad = h + (\epsilon - h) = \epsilon \end{aligned}$$

and we can conclude that $(\bar{y}_{1,n}, \bar{y}_{2,n}) \in \tilde{S}(x_n, \epsilon)$ with $\bar{y}_{1,n} \to y_1$ and $\bar{y}_{2,n} \to y_2$.

Remark 3.3. Assumptions (3.6) (respectively (3.7)) can be weakened assuming that $y_2 \in Y_2$ is such that $\{y_1 \in Y_1 : (y_1, y_2) \in \tilde{S}(x, \epsilon)\} \neq \emptyset$ (respectively $y_1 \in Y_1$ is such that $\{y_2 \in Y_2 : (y_1, y_2) \in \tilde{S}(x, \epsilon)\} \neq \emptyset$).

By strengthening assumptions (3.6) and (3.7) we can obtain $\tilde{S}(., \epsilon)$ open graph.

PROPOSITION 3.3. *Let $x \in X$. If the following assumptions are satisfied,*

$$\begin{cases} \text{for any sequence } (x_n) \text{ converging to } x, \text{ any } y_2 \in Y_2 \\ \text{any sequence } (y_{2,n}) \text{ converging to } y_2 \text{ and any sequence } (y_{1,n}) \text{ in } Y_1 \\ \text{there exists } \bar{y}_1 \in Y_1 \text{ such that} \\ \liminf_{n \to \infty} f(x_n, y_{1,n}, y_{2,n}) \geq f(x, \bar{y}_1, y_2) \end{cases} \tag{3.10}$$

$$\begin{cases} \text{for any sequence } (x_n) \text{ converging to } x, \text{ any } y_1 \in Y_1 \\ \text{any sequence } (y_{1,n}) \text{ converging to } y_1 \text{ and any sequence } (y_{2,n}) \text{ in } Y_2 \\ \text{there exists } \bar{y}_2 \in Y_2 \text{ such that} \\ \limsup_{n \to \infty} f(x_n, y_{1,n}, y_{2,n}) \leq f(x, y_1, \bar{y}_2), \end{cases} \tag{3.11}$$

then the multifunction $\tilde{S}(., \epsilon)$ is sequentially open graph at x for any $\epsilon > 0$.

Proof. Let (x_n) converging to x, $(y_1, y_2) \in \tilde{S}(x, \epsilon)$, and $(y_{1,n}, y_{2,n})$ converging to (y_1, y_2). We have $v_2(x, y_1) - v_1(x, y_2) < \epsilon$ but assumption (3.10) (respectively (3.11)) guarantees that the function v_1 (respectively v_2) is sequentially lower semicontinuous (respectively upper semicontinuous) then

$$\limsup_{n \to \infty} v_2(x_n, y_{1,n}) - \liminf_{n \to \infty} v_1(x_n, y_{2,n}) < \epsilon$$

and, proceeding as in the proof of Proposition 3.2, we can conclude that, for n sufficiently large, $(y_{1,n}, y_{2,n}) \in \tilde{S}(x_n, \epsilon)$.

Remark 3.4. Let us note that the assumption (3.10) (respectively (3.11)) is satisfied if f is sequentially lower semicontinuous (respectively sequentially upper semicontinuous) at (x, y_1, y_2) for any $(y_1, y_2) \in Y_1 \times Y_2$ and Y_1 (respectively Y_2) is sequentially compact but the contrary is not true as shown by the following example.

EXAMPLE 3.2. Let $X = Y_1 = Y_2 = [-1, 1]$ and f: $X \times Y_1 \times Y_2 \to R$ defined by

$$f(x, y_1, y_2) = \begin{cases} y_1 + y_2 & \text{if } (y_1, y_2) \in Y_1 \times Y_2 - \{(0,0)\} \\ 2 & \text{if } (y_1, y_2) = (0,0) \end{cases}$$

for any $x \in X$.

Such a function is not lower semicontinuous at $(x, 0, 0)$ for any $x \in X$ but satisfies assumption (3.10). In fact if $y_2 = 0$ and $(y_{2,n})$ is a sequence converging to 0 in Y_2 for any sequence $(y_{1,n})$ in Y_1 we have

$$\liminf_{n \to \infty} f(x_n, y_{1,n}, y_{2,n}) \geq -1 = f(x, -1, 0).$$

Finally let us study the lower semicontinuity of the multifunction $S(., \epsilon)$ for $\epsilon > 0$.

PROPOSITION 3.4. *Let Y_1 and Y_2 be two first countable topological convex spaces and $x \in X$. If the following assumptions are satisfied*: (3.6), (3.7), *and*

$$\text{the function } f(x, ., y_2) \text{ is convex on } Y_1 \text{ for any } y_2 \in Y_2 \tag{3.12}$$

$$\text{the function } f(x, y_1, .) \text{ is concave on } Y_2 \text{ for any } y_2 \in Y_2, \tag{3.13}$$

then the multifunction $S(., \epsilon)$ is sequentially lower semicontinuous at x for any $\epsilon > 0$.

Proof. For any $\epsilon > 0$ the subset $\tilde{S}(x, \epsilon)$ is non-empty because the function $f(x, ., .)$ is supposed to have a saddle value. Now let us prove that $S(x, \epsilon) \subseteq \operatorname{cl} \tilde{S}(x, \epsilon)$ where cl A denotes the closure of the set A. Consider $(\bar{y}_1, \bar{y}_2) \in S(x, \epsilon)$, $(\tilde{y}_1, \tilde{y}_2) \in \tilde{S}(x, \epsilon)$, and for any $n \in N$,

$$\bar{y}_{1,n} = \frac{1}{n}\tilde{y}_1 + \left[1 - \frac{1}{n}\right]\bar{y}_1 \quad \text{and} \quad \bar{y}_{2,n} = \frac{1}{n}\tilde{y}_2 + \left[1 - \frac{1}{n}\right]\bar{y}_2.$$

As in [14, Proposition 6.2], we can prove that $(\bar{y}_{1,n}, \bar{y}_{2,n})$ is convergent to $(\bar{y}_1, \bar{y}_2)$. Moreover, from (3.12), we have

$$\begin{aligned}
\sup_{y_2 \in Y_2} f(x, \bar{y}_{1,n}, y_2) &\le \sup_{y_2 \in Y_2} \left[\frac{1}{n} f(x, \tilde{y}_1, y_2) + \left(1 - \frac{1}{n}\right) f(x, \bar{y}_1, y_2) \right] \\
&\le \frac{1}{n} \sup_{y_2 \in Y_2} f(x, \tilde{y}_1, y_2) + \left(1 - \frac{1}{n}\right) \sup_{y_2 \in Y_2} f(x, \bar{y}_1, y_2) \\
&= \frac{1}{n} v_2(x, \tilde{y}_1) + \left(1 - \frac{1}{n}\right) v_2(x, \bar{y}_1).
\end{aligned}$$

Similarly, by using (3.13), we can deduce

$$\inf_{y_1 \in Y_1} f\left(x, y_1, \frac{1}{n}\tilde{y}_2 + \left(1 - \frac{1}{n}\right)\bar{y}_2\right) \ge \frac{1}{n} v_1(x, \tilde{y}_2) + \left(1 - \frac{1}{n}\right) v_1(x, \bar{y}_2)$$

then

$$\begin{aligned}
&v_2(x, \bar{y}_{1,n}) - v_1(x, \bar{y}_{2,n}) \\
&\quad \le \frac{1}{n}\left[v_2(x, \tilde{y}_1) - v_1(x, \tilde{y}_2)\right] + \left(1 - \frac{1}{n}\right)\left[v_2(x, \bar{y}_1) - v_1(x, \bar{y}_2)\right] \\
&\quad < \epsilon
\end{aligned}$$

and $(\bar{y}_{1,n}, \bar{y}_{2,n}) \in \tilde{S}(x, \epsilon)$. But $(\bar{y}_{1,n}, \bar{y}_{2,n})$ is convergent to $(\bar{y}_1, \bar{y}_2)$ and then $(\bar{y}_1, \bar{y}_2) \in \operatorname{cl} \tilde{S}(x, \epsilon)$.

Finally, let $\epsilon > 0$ and (x_n) be a sequence converging to x. From Proposition 3.2 we have

$$S(x, \epsilon) \subseteq \operatorname{cl} \tilde{S}(x, \epsilon) \subseteq \operatorname{cl} \operatorname*{Lim\,inf}_n \tilde{S}(x_n, \epsilon).$$

But Y_1 and Y_2 being first countable topological spaces, $\operatorname*{Lim\,inf}_n \tilde{S}(x_n, \epsilon)$ is a sequentially closed subset in $Y_1 \times Y_2$ and

$$S(x, \epsilon) \subseteq \operatorname*{Lim\,inf}_n \tilde{S}(x_n, \epsilon) \subseteq \operatorname*{Lim\,inf}_n S(x_n, \epsilon).$$

Remark 3.5. If Y_1 and Y_2 are not first countable topological spaces it is possible to obtain $S(., \epsilon)$ nearly sequentially lower semicontinuous in x, that is [11],

$$S(x, \epsilon) \subseteq \overline{\operatorname*{Lim\,inf}_n S(x_n, \epsilon)}^{s},$$

where $\overline{A}^{s}$ is the sequential closure of A.

Remark 3.6. In [7] under assumptions on the marginal functions v_1 and v_2 the following result for $S(., \epsilon)$ has been obtained: for any sequence (x_n) converging to x and for any $(y_1, y_2) \in S(x, \epsilon)$ there exist a sequence $(y_{1,n}, y_{2,n})$ converging to (y_1, y_2) and a sequence (ϵ_n) converging to ϵ (with $\epsilon_n > \epsilon$) such that $(y_{1,n}, y_{2,n}) \in S(x_n, \epsilon_n)$ for n large enough.

Note that $\epsilon_n > \epsilon$ and ϵ_n depends on x_n, x, y_1, y_2 so this property does not imply the lower semicontinuity of the multifunction $S(., \epsilon)$ at x.

4. SADDLE POINT UNDER CROSSED CONSTRAINTS

In this section we consider the case in which the strategy of a player belongs to a subset depending not only on the parameter x but also on the strategy of the other player. Now we are interested in the multifunctions $S(., \epsilon)$ and $\tilde{S}(x, \epsilon)$ as defined in Definition 2.4 and the marginal functions v_1 and v_2 are defined by (2.1).

A first result is the following:

PROPOSITION 4.1. *Let $x \in X$. If the following assumptions are satisfied,*

$$\begin{cases} \text{the function } f \text{ is sequentially upper semicontinuous at } (x, y_1, y_2) \\ \text{for any } x \in X,\ y_2 \in Y_2, \text{ and } y_1 \in K_1(x, y_2); \end{cases} \tag{4.1}$$

$$\begin{cases} \text{the function } f \text{ is sequentially lower semicontinuous at } (x, y_1, y_2) \\ \text{for any } x \in X,\ y_1 \in Y_1, \text{ and } y_2 \in K_2(x, y_1); \end{cases} \tag{4.2}$$

$$\begin{cases} \text{the multifunctions } K_1 \text{ and } K_2 \text{ are sequentially closed graph} \\ \text{respectively at } (x, y_2) \text{ for any } y_2 \in Y_2 \text{ and at } (x, y_1) \text{ for any } y_1 \in Y_1; \end{cases} \tag{4.3}$$

$$\begin{cases} \text{the multifunctions } K_1 \text{ and } K_2 \text{ are sequentially lower semicontinuous} \\ \text{respectively at } (x, y_2) \text{ for any } y_2 \in Y_2 \text{ and at } (x, y_1) \text{ for any } y_1 \in Y_1; \end{cases} \tag{4.4}$$

the function f is marginally proper with respect to the constraints, that is,

$$\begin{cases} \text{for any } x \in X, \text{ for any } y_2 \in Y_2 \text{ there exists } \bar{y}_1 \in K_1(x, y_2) \text{ such that} \\ f(x, \bar{y}_1, y_2) < +\infty; \end{cases} \tag{4.5}$$

$$\begin{cases} \text{for any } x \in X, \text{ for any } y_1 \in Y_1 \text{ there exists } \bar{y}_2 \in K_2(x, y_1) \text{ such that} \\ f(x, y_1, \bar{y}_2) > -\infty; \end{cases} \tag{4.6}$$

then the multifunction $S(., \epsilon)$ is sequentially closed graph at x for any $\epsilon \geq 0$.

Proof. Under assumptions (4.1) and (4.4) (respectively (4.2) and (4.4)) the marginal function v_1 is sequentially upper semicontinuous at (x, y_2) for any $y_2 \in Y_2$ (respectively v_2 is sequentially lower semicontinuous at (x, y_1) for any $y_1 \in Y_1$) (see [12, Proposition 3.2.1]). Let (x_n) converge to x and $(\bar{y}_1, \bar{y}_2) \in Y_1 \times Y_2$ such that there exist two sequences $(y_{1,k})$ and $(y_{2,k})$ respectively convergent to $\bar{y}_1$ and $\bar{y}_2$ and satisfying $(y_{1,k}, y_{2,k}) \in S(x_{n_k}, \epsilon)$ with $y_{1,k} \in K_1(x_{n_k}, y_{2,k})$ and $y_{2,k} \in K_2(x_{n_k}, y_{1,k})$. As in the proof of Proposition 3.1 we can prove that $v_2(x, \bar{y}_1) - v_1(x, \bar{y}_2) \le \epsilon$. But $(y_{1,k})$ and $(y_{2,k})$ are respectively convergent to $\bar{y}_1$ and $\bar{y}_2$ and, the multifunctions K_1 and K_2 being sequentially closed graph, $\bar{y} \in K_1(x, \bar{y}_2)$, $\bar{y}_2 \in K_2(x, \bar{y}_1)$, and $(\bar{y}_1, \bar{y}_2) \in S(x, \epsilon)$ for any $\epsilon \ge 0$.

Now, in order to study lower semicontinuity in this case, for any $x \in X$ let $(K_1 \otimes K_2)(x)$: $Y_1 \times Y_2 \to Y_1 \times Y_2$ be the multifunction defined by

$$[(K_1 \otimes K_2)(x)](y_1, y_2) = K_1(x, y_2) \times K_2(x, y_1). \tag{4.7}$$

Let $\mathrm{Fix}[(K_1 \otimes K_2)(x)]$ be the set of fixed points of $(K_1 \otimes K_2)(x)$ and $\mathrm{Fix}\, K_1 \otimes K_2$: $X \to Y_1 \times Y_2$ be the multifunction defined by

$$(\mathrm{Fix}\, K_1 \otimes K_2)(x) = \mathrm{Fix}[(K_1 \otimes K_2)(x)]. \tag{4.8}$$

DEFINITION 4.1. The pair (K_1, K_2) is sequentially cross lower semicontinuous at x iff the multifunction $\mathrm{Fix}\, K_1 \otimes K_2$ (defined by (4.8)) is sequentially lower semicontinuous at x, that is, for any sequence (x_n) converging to x and any $(y_1, y_2) \in Y_1 \times Y_2$ such that $y_1 \in K_1(x, y_2)$ and $y_2 \in K_2(x, y_1)$ there exist two sequences $(\bar{y}_{1,n})$ and $(\bar{y}_{2,n})$ respectively converging to y_1 and y_2 such that, for any $n \in N$, $\bar{y}_{1,n} \in K_1(x_n, \bar{y}_{2,n})$ and $\bar{y}_{2,n} \in K_2(x_n, \bar{y}_{1,n})$.

DEFINITION 4.2. The pair (K_1, K_2) is sequentially cross open graph at x iff the multifunction $\mathrm{Fix}\, K_1 \otimes K_2$ is sequentially open graph at x, that is, for any sequence (x_n) converging to x, any $(y_1, y_2) \in Y_1 \times Y_2$ such that $y_1 \in K_1(x, y_2)$ and $y_2 \in K_2(x, y_1)$, any sequence $(y_{1,n})$ converging to y_1 and any sequence $(y_{2,n})$ converging to y_2, we have, for n sufficiently large, $y_{1,n} \in K_1(x_n, y_{2,n})$ and $y_{2,n} \in K_2(x_n, y_{1,n})$.

DEFINITION 4.3. The pair (K_1, K_2) is cross convex-valued iff the multifunction $\mathrm{Fix}\, K_1 \otimes K_2$ is convex-valued.

Now we are able to study the properties of $\tilde{S}(., \epsilon)$ and $S(., \epsilon)$ for $\epsilon > 0$.

PROPOSITION 4.2. *Let $x \in X$. Assume that* (4.1), (4.2), (4.4), *and the following assumption are satisfied*:

$$\textit{the pair } (K_1, K_2) \textit{ is sequentially cross lower semicontinuous at } x. \tag{4.9}$$

Then the multifunction $\tilde{S}(., \epsilon)$ is sequentially lower semicontinuous at x for any $\epsilon > 0$.

Proof. From Proposition 3.2.1 in [12] the function v_1 is sequentially upper semicontinuous at (x, y_2) for any $y_2 \in Y_2$ and the function v_2 is sequentially lower semicontinuous at (x, y_1) for any $y_1 \in Y_1$. Let (x_n) be a sequence converging to x and $(y_1, y_2) \in \tilde{S}(x, \epsilon)$. From (4.9) there exist two sequences $(\bar{y}_{1,n})$ and $(\bar{y}_{2,n})$ respectively converging to y_1 and y_2 such that for any $n \in N$, $\bar{y}_{1,n} \in K_1(x_n, \bar{y}_{2,n})$ and $\bar{y}_{2,n} \in K_2(x_n, \bar{y}_{1,n})$ therefore, by proceeding as in Proposition 3.2, we can prove that $v_2(x_n, \bar{y}_{1,n}) - v_1(x_n, \bar{y}_{2,n}) < \epsilon$ for n sufficiently large, that is, $(\bar{y}_{1,n}, \bar{y}_{2,n}) \in \tilde{S}(x_n, \epsilon)$.

Analogously we can easily prove:

PROPOSITION 4.3. *Let $x \in X$. If assumptions* (4.1), (4.2), (4.4), *and the following are satisfied,*

$$\textit{the pair } (K_1, K_2) \textit{ is sequentially cross open graph at } x, \tag{4.10}$$

then the multifunction $\tilde{S}(., \epsilon)$ is sequentially open graph at x for any $\epsilon > 0$.

So we obtain the following result:

PROPOSITION 4.4. *Let Y_1 and Y_2 be two first countable topological convex spaces, $x \in X$, and $\epsilon > 0$. If the assumptions* (4.1), (4.2), (4.4), (4.9), (3.12), *and* (3.13) *are satisfied and*

$$\textit{the pair } (K_1, K_2) \textit{ is cross convex-valued at } x \tag{4.11}$$

then the multifunction $S(., \epsilon)$ is sequentially lower semicontinuous at x.

Proof. Prove that $S(x, \epsilon) \subseteq \operatorname{cl} \tilde{S}(x, \epsilon)$. Let $(\tilde{y}_1, \tilde{y}_2) \in \tilde{S}(x, \epsilon) \neq \emptyset$, $(\bar{y}_1, \bar{y}_2) \in S(x, \epsilon)$, and $\bar{y}_{1,n} = (1/n)\tilde{y}_1 + (1 - (1/n))\bar{y}_1$, $\bar{y}_{2,n} = (1/n)\tilde{y}_2 + (1 - (1/n))\bar{y}_2$. Then $(\bar{y}_1, \bar{y}_2) \in \operatorname{Fix}(K_1 \otimes K_2)(x)$ and $(\tilde{y}_1, \tilde{y}_2) \in \operatorname{Fix}(K_1 \otimes K_2)(x)$.

So, from assumption (4.11), $(\bar{y}_{1,n}, \bar{y}_{2,n}) = (1/n)(\tilde{y}_1, \tilde{y}_2) + (1 - 1/n)(\bar{y}_1, \bar{y}_2) \in [\operatorname{Fix}(K_1 \otimes K_2)](x)$ and $\bar{y}_{1,n} \in K_1(x, \bar{y}_{2,n})$, $\bar{y}_{2,n} \in K_2(x, \bar{y}_{1,n})$.

In order to conclude, it is now sufficient to proceed as in Proposition 3.4.

Remark 4.1. If Y_1 and Y_2 are not first countable topological spaces it is possible to obtain $S(., \epsilon)$ nearly sequentially lower semicontinuous at x.

Now let us consider the case in which the multifunctions K_1 and K_2 are defined by a finite number of inequalities, that is,

$$K_1(x, y_2) = \{y_1 \in Y_1 \text{ such that } g_{1,j}(x, y_1, y_2) \leq 0 \text{ for } j = 1, \dots, q\};$$

$$K_2(x, y_1) = \{y_2 \in Y_2 \text{ such that } g_{2,h}(x, y_1, y_2) \leq 0 \text{ for } h = 1, \dots, p\},$$

where $g_{1,j}$ and $g_{2,h}$ are real valued functions defined on $X \times Y_1 \times Y_2$. In this case it is possible to express conditions (4.4) and (4.9) in terms of the functions $g_{1,j}$ and $g_{2,h}$. In order to simplify, here we consider only the case in which $q = p = 1$ and we set $g_{1,1} = g_1$, $g_{2,1} = g_2$.

PROPOSITION 4.5. *Let $x \in X$. If the following assumptions are satisfied*: (4.1), (4.2), *and*

$$Y_1 = R^n, \qquad Y_2 = R^m \qquad \textit{with } n, m \in N \tag{4.12}$$

$$\begin{cases} \textit{the functions } g_1(x,.,.) \textit{ and } g_2(x,.,.) \\ \textit{are strictly quasi-convex in } Y_1 \times Y_2, \textit{ that is } [17], \\ g_i(x, ty_1 + (1-t)z_1, ty_2 + (1-t)z_2) < \\ \max\{g_i(x, y_1, z_1), g_i(x, y_2, z_2)\} \\ \textit{for any } t \in]0,1[, \textit{ for } i = 1,2 \\ \textit{and for any } (y_1, z_1), (y_2, z_2) \in Y_1 \times Y_2 \\ \textit{verifying } g_i(x, y_1, z_1) \neq g_i(x, y_2, z_2); \end{cases} \tag{4.13}$$

$$\begin{cases} \textit{there exists } (y_1, y_2) \in Y_1 \times Y_2 \textit{ such that} \\ g_1(x, y_1, y_2) < 0 \textit{ and } g_2(x, y_1, y_2) < 0; \end{cases} \tag{4.14}$$

$$\begin{cases} \textit{the functions } g_1(x,.,.) \textit{ and } g_2(x,.,.) \textit{ are upper} \\ \textit{semicontinuous at } Y_1 \times Y_2; \end{cases} \tag{4.15}$$

$$\begin{cases} \textit{for any } y_2 \in Y_2 \textit{ there exists } y_1 \in Y_1 \textit{ such that } g_1(x, y_1, y_2) \leq 0 \\ \textit{and for any } y_1 \in Y_1 \textit{ there exists } y_2 \in Y_2 \textit{ such that } g(x, y_1, y_2) \leq 0; \end{cases} \tag{4.16}$$

$$\begin{cases} \textit{for any sequence } (x_n) \textit{ converging to } x, \textit{ any } (y_1, y_2) \in Y_1 \times Y_2, \\ \textit{any sequence } (y_{2,n}) \ (\textit{respectively } (y_{1,n})) \textit{ converging to } y_2 \\ (\textit{respectively } y_1) \textit{ there exists a sequence } (y_{1,n}) \ (\textit{respectively } (y_{2,n})) \\ \textit{converging to } y_1 \ (\textit{respectively } y_2) \textit{ such that} \\ \limsup_{n \to \infty} f(x_n, y_{1,n}, y_{2,n}) \leq f(x, y_1, y_2), \end{cases} \tag{4.17}$$

then the multifunction $\tilde{S}(., \epsilon)$ is sequentially lower semicontinuous at x for any $\epsilon > 0$.

Proof. From assumptions (4.12), (4.13), (4.16), and (4.17), condition (4.4) is satisfied (see Proposition 3.3 and Remark 3.1 in [13]). Moreover let H_1 and H_2 be the two multifunctions defined on X, and valued in $Y_1 \times Y_2$, by

$$H_i(x) = \{(y_1, y_2) \in Y_1 \times Y_2 \text{ such that } g_i(x, y_1, y_2) \leq 0\} \qquad \text{for } i \in \{1, 2\}.$$

It is easy to verify that (4.9) and

$$H_1 \cap H_2 \text{ is sequentially lower semicontinuous at } x \tag{4.18}$$

are equivalent conditions.

So from assumptions (4.12), (4.13), (4.14), (4.15), and (4.17) we deduce that the condition (4.18) is satisfied (see Proposition 3.4 and Remark 3.1 in [13]) and it is possible to apply Proposition 4.2.

For what concerns the multifunction $S(., \epsilon)$ we obtain:

PROPOSITION 4.6. *Let $x \in X$. If the following assumptions are satisfied*: (3.12), (3.13), (4.12) *to* (4.17), *and*

$$\textit{the functions } g_1(x,.,.) \textit{ and } g_2(x,.,.) \textit{ are quasi-convex on } Y_1 \times Y_2, \tag{4.19}$$

then the multifunction $S(., \epsilon)$ is sequentially lower semicontinuous at x for any $\epsilon > 0$.

Proof. From the proof of Proposition 4.5 we know that conditions (4.4) and (4.9) are satisfied. From condition (4.19) we deduce that (4.11) is satisfied and we can apply Proposition 4.4.

Remark 4.2. Let us recall that a strictly quasi-convex function is not necessarily quasi-convex but a strictly quasi-convex and lower semicontinuous function is also quasi-convex (see [17]).

5. EXISTENCE AND APPROXIMATION RESULTS FOR WEAK HIERARCHICAL SADDLE POINT PROBLEMS

In order to give an example of an application of the previous results we consider now the weak hierarchical saddle point problem (w-HSPP) defined in the Introduction,

$$\text{(w-HSPP)} \quad \begin{cases} \text{find } \bar{x} \in X \text{ such that} \\ \inf\limits_{x \in X} \sup\limits_{(y_1, y_2) \in S(x)} l(x, y_1, y_2) = \sup\limits_{(y_1, y_2) \in S(\bar{x})} l(\bar{x}, y_1, y_2) \end{cases}$$

and we suppose that X, Y_1, Y_2 are three sequentially compact spaces.

To obtain existence of solutions to (w-HSPP) it is sufficient to have lower semicontinuity of the marginal function w defined by

$$w(x) = \sup_{(y_1, y_2) \in S(x)} l(x, y_1, y_2).$$

But the lower semicontinuity of S is a sufficient condition for the lower semicontinuity of w (see, for example, [12, Proposition 3.2.1]) so we consider the regularized problem,

$$(\text{w-HSPP})(\epsilon) \begin{cases} \text{find } \bar{x} \in X \text{ such that} \\ \inf_{x \in X} \sup_{(y_1, y_2) \in S(x, \epsilon)} l(x, y_1, y_2) = \sup_{(y_1, y_2) \in S(\bar{x}, \epsilon)} l(\bar{x}, y_1, y_2), \end{cases}$$

where $S(x, \epsilon)$ is the set of ϵ-saddle points for the problem $S(x)$ as defined in Definition 2.4. We obtain:

THEOREM 5.1. *If the assumptions of Proposition* 4.4 (*resp. Proposition* 4.6 *when the multifunctions K_1 and K_2 are defined by inequalities*) *and the following are satisfied,*

$$\textit{the function } l \textit{ is sequentially lower semicontinuous at } X \times Y_1 \times Y_2, \tag{5.1}$$

then there exists a solution to (*w-HSPP*)(ϵ).

Proof. In our assumptions, the multifunction $S(., \epsilon)$ is lower semicontinuity so to conclude it takes only to apply Proposition 2.3.1 of [11].

Now let

$$v(\epsilon) = \inf_{x \in X} \sup_{(y_1, y_2) \in S(x, \epsilon)} l(x, y_1, y_2)$$

for any $\epsilon \geq 0$ and $v(0) = v$ then we have:

THEOREM 5.2. *Assume that the assumptions of Theorem* 5.1 *and the following are satisfied*:

$$\begin{cases} \textit{for any } x \in X, \textit{ there exists a sequence } (x_n) \textit{ converging to } x \\ \textit{such that, for any } (y_1, y_2) \in Y_1 \times Y_2 \textit{ and any sequence } (y_{1,n}, y_{2,n}) \\ \textit{converging to } (y_1, y_2) \textit{ we have } \limsup_{n \to \infty} l(x_n, y_{1,n}, y_{2,n}) \leq l(x, y_1, y_2). \end{cases} \tag{5.2}$$

Then $\lim_{\epsilon \to 0} v(\epsilon) = v$.

Proof. Let (ϵ_n) be a sequence of real positive numbers such that $\lim_{n \to \infty} \epsilon_n = 0$. We can prove that

$$\lim_{n \to \infty} v(\epsilon_n) = v. \tag{5.3}$$

In fact, from $S(x) \subseteq S(x, \epsilon)$, we get

$$v \leq \liminf_{n \to \infty} v(\epsilon_n) \tag{5.4}$$

and, proceeding as in [15, Proposition 2.4] we can prove

$$\limsup_{n \to \infty} v(\epsilon_n) \leq v. \tag{5.5}$$

From (5.4) and (5.5) we have (5.3) and now it is easy to conclude.

REFERENCES

1. H. Attouch and J. B. Wets, A convergence theory for saddle functions, *Trans. Amer. Math. Soc.* **280** (1983).
2. J. P. Aubin and H. Frankowska, "Set-Valued Analysis," Systems & Control: Foundations & Applications, Birkhäuser, Basel, 1990.
3. T. Basar and G. J. Olsder, "Dynamic Noncooperative Game Theory," Academic Press, New York, 1982.
4. E. Cavazzuti, Γ-convergenze multiple, convergenze di punti sella e di maxmin, *Boll. Un. Mat. Ital. B* **1** (1982), 251–274.
5. E. Cavazzuti and J. Morgan, Well-posed saddle point problems, *in* "Optimization, Theory and Algorithms" (J. B. Hiriart-Urruty, W. Oettli, and J. Stoer, Eds.), pp. 61–76, Dekker, New York, 1983.
6. M. C. Delfour and J. Morgan, On one sided derivative of minmax and saddle points with respect to a parameter, *Optimization* **31** (1994), 343–358.
7. J. Guillerme, Convergence of approximate saddle point, *J. Math. Anal. Appl.* **137** (1989), 297–311.
8. P. T. Harker and J. S. Pang, Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications, *Math. Programming* **48** (1990), 161–220.
9. K. Kuratowski, "Topology," Academic Press, New York, 1966.
10. G. Leitmann, On generalized Stackelberg strategies, *J. Optim. Theory Appl.* **26** (1978).
11. M. B. Lignola and J. Morgan, Existence and approximation for min-sup problems, *in* "Proceedings of the International Conference on Operation Research 90 in Vienna" (G. Feichtinger, W. Buhler, F. J. Radermacher, and P. Stahly, Eds.), pp. 157–164, Springer-Verlag, Berlin, 1992.
12. M. B. Lignola and J. Morgan, Semicontinuities of marginal functions in a sequential setting, *Optimization* **24** (1992), 307–318.
13. M. B. Lignola and J. Morgan, Topological existence and stability for Stackelberg problems, *J. Optim. Theory Appl.* **84** (1995), 145–169.
14. P. Loridan and J. Morgan, New results on approximate solutions in two-level optimization, *Optimization* **20** (1989), 819–836.
15. P. Loridan and J. Morgan, On strict ϵ-solutions for a two level optimization problem, *in* "Proceedings of the International Conference on Operation Research 90 in Vienna" (W. Buhler, G. Feichtinger, F. Harti, F. J. Radermacher, and P. Stahly, Eds.), pp. 165–172, Springer-Verlag, Berlin, 1992.
16. L. Mac-Linden, A minimax theorem, *Math. Oper. Res.* **9** (1984), 576–591.
17. O. L. Mangasarian, "Non Linear Programming," McGraw–Hill, New York, 1969.